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Shallow water wave generation by unsteady flow

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Small amplitude waves on a shallow layer of water are studied from the point of view used in aerodynamic sound theory. It is shown that many aspects of the generation and propagation of water waves are similar to those of sound waves in air. Certain differences are also discussed. It is concluded that shallow water simulation can be employed in the study of some aspects of aerodynamically generated sound.

1. Introduction

The theory of aerodynamic sound initiated by Lighthill is built upon the equations of mass and momentum conservation. These yield a three-dimensional wave equation which describes the generation and propagation of sound waves. It is shown in this paper that the generation and propagation of waves on a shallow layer of water is governed by a two-dimensional wave equation similar to Lighthill's and that, in many respects, these waves behave like sound waves. This similarity enables sound waves to be simulated and visualized in the laboratory. The ability to see the waves makes it possible to study their development and interactions in detail, a study which would be extremely difficult with aerial sound waves. Furthermore, since the propagation speed of water waves is small, high Mach number situations can be examined easily.

Gravity waves on a finite depth of water are dispersive and cannot in general provide a very good model for sound waves, which are non-dispersive. However, at a particular mean depth (0·5 cm), surface tension effects render the water layer practically non-dispersive, thus minimizing this difficulty. Indeed waves on a shallow water layer are an excellent simulation of two-dimensional aerial waves, and they also share many outstanding features in common with three-dimensional waves. The similarity is brought out in detail in the following analysis which considers turbulence as a source of shallow water waves. The philosophy underlying the derivation of the equations is the same as that used in the sound theory (Lighthill 1952). The waves are regarded as a by-product of a more complicated flow, and the problem is to estimate the waves generated by it. The flow is assumed to be known, and acts as a source of waves which radiate into the undisturbed water. From this point of view, the resultant forced wave equation, although rather artificially manufactured, is a correct description of the field.

The general solution of the shallow water equation demonstrates that water and sound waves are alike in that both are generated by the same distribution of sources, and that at large distances from the sources both are waves of constant profile radiating out at the constant wave speed. Of course, to be energy conserving in two dimensions, the amplitude of the water waves falls off only as the square root of the radiation distance, but this difference is minor. A more important difference is that the water wave amplitude depends upon a time integral of the source strength, a result not found in three dimensions. Consequently, this amplitude has a dimensional dependence different from that of sound waves.

The shallow water equations are derived and written as an inhomogeneous wave equation in §2. In §3, this equation is solved in the sense of Curle (1955). That is, the radiation field is given explicitly in terms of a known distribution of surface quadrupoles and a line distribution of dipoles, whose strength involves the field quantity. The quadrupole terms are regarded as specified in a moving reference frame in §4. The convective effects are described and the lack of a distinct singularity at the Mach wave condition is accounted for as being an essential difference between the two- and three-dimensional theories. The paper is concluded with a brief summary of the similarities that exist between aerial sound waves and shallow water waves and discusses the possibility for effective similation of aerodynamically generated sound on a shallow water table.

2. Equations of motion

Before the equations of motion can be derived, we must demonstrate that a particular depth exists at which surface tension effects render the water layer a practically non-dispersive medium. To do this, we examine the formula for the wave speed c of small amplitude waves of wave-number k. This is given by Milne-Thomson (1960, p. 409) in the form

$$c^2 = \left(\frac{g}{k}+\frac{Sk}{\rho_w}\right)\tanh kh_0, \tag{2.1}$$

ρ_w is the density of the water, S the surface tension coefficient, and h_0 the mean depth. For small kh_0, this formula can be expanded in a rapidly converging power series,

$$c^2 = gh_0+\left(\frac{S}{\rho_w h_0}-\frac{gh_0}{3}\right)(kh_0)^2+O(kh_0)^4. \tag{2.2}$$

In the absence of surface tension, the variation of c^2 from its zero wave-number value gh_0 is of order $(kh_0)^2$, but with surface tension, h_0 can be chosen so that

$$\frac{S}{\rho_w h_0}-\frac{gh_0}{3}$$

is zero, leaving an error of order $(kh_0)^4$. For this critical depth, the wave speed is constant for all but the shortest waves, and a wave equation with constant c correctly describes the motion. For water at room temperature this critical depth is 0·48 cm, and all references to shallow water refer to this depth.

The appropriate form of the equations of motion are derived by integrating the usual equations vertically through the water layer, whose variable depth is denoted by h. The water is assumed to be inviscid. The subscripts α, β, γ range over the values 1, 2 (the two horizontal directions), and repeated subscripts imply a tensor summation over these values. Small letters denote quantities at a point, whereas capital letters denote the average value through the depth, e.g. the average velocity U_α is

$$U_\alpha = \frac{1}{h}\int_0^h u_\alpha dx_3. \tag{2.3}$$

Consider first the integrated continuity equation

$$\int_0^h \left(\frac{\partial u_\alpha}{\partial x_\alpha} + \frac{\partial u_3}{\partial x_3}\right) dx_3 = 0,$$

or

$$\frac{\partial}{\partial x_\alpha}\int_0^h u_\alpha dx_3 - \frac{\partial h}{\partial x_\alpha}[u_\alpha]_{x_3=h} + [u_3]_{x_3=h} = 0, \tag{2.4}$$

as $[u_3]_{x_3=0}$ is zero. $[u_3]_{x_3=h}$ is the particle velocity at the surface, so that it is equal to $[Dh/Dt]_{x_3=h}$, or

$$[u_3]_{x_3=h} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x_\alpha}[u_\alpha]_{x_3=h}. \tag{2.5}$$

These equations combine to give the mean form of the continuity equation,

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x_\alpha}(hU_\alpha) = 0. \tag{2.6}$$

We similarly integrate the α-component of the momentum equation

$$\int_0^h \left(\frac{\partial u_\alpha}{\partial t} + u_\beta \frac{\partial u_\alpha}{\partial x_\beta} + u_3 \frac{\partial u_\alpha}{\partial x_3}\right) dx_3 = -\int_0^h \frac{1}{\rho_w}\frac{\partial p}{\partial x_\alpha} dx_3,$$

which leads to the result,

$$\frac{\partial}{\partial t}(hU_\alpha) + \frac{\partial}{\partial x_\beta}(hU_\alpha U_\beta) = -\int_0^h \frac{1}{\rho_w}\frac{\partial p}{\partial x_\alpha} dx_3. \tag{2.7}$$

$hU_\alpha U_\beta$ has been written for

$$\int_0^h u_\alpha u_\beta dx_3,$$

although notationally this is not strictly consistent. The vertical momentum equation is used to eliminate the pressure from (2.7). As

$$\frac{Du_3}{Dt} = -g - \frac{1}{\rho_w}\frac{\partial p}{\partial x_3},$$

then

$$\frac{1}{\rho_w}(p-p_s) = g(h-x_3) + \int_{x_3}^h \frac{Du_3}{Dt} dx_3', \tag{2.8}$$

where p_s is the pressure in the water just below the surface, which, owing to the surface tension, differs from the atmospheric pressure p_a. This difference is given by the Laplace formula, and for surfaces which deviate only slightly from a plane (i.e. the amplitude of the displacement is small compared with the horizontal length scale), can be written (Landau & Lifshitz 1959, p. 233)

$$p_a - p_s = S\frac{\partial^2 h}{\partial x_\alpha^2}. \tag{2.9}$$

Substitution of (2.8) and (2.9) into (2.7) gives

$$\left.\begin{aligned} \frac{\partial}{\partial t}(hU_\alpha)+\frac{\partial}{\partial x_\beta}(hU_\alpha U_\beta) &= -gh\frac{\partial h}{\partial x_\alpha}+\frac{Sh}{\rho_w}\frac{\partial^3 h}{\partial x_\alpha \partial x_\gamma \partial x_\gamma}-\frac{\partial B}{\partial x_\alpha}, \\ \text{in which}\qquad B &= \int_0^h dx_3\int_{x_3}^h \frac{Du_3}{Dt}dx_3'. \end{aligned}\right\} \tag{2.10}$$

Now
$$h\frac{\partial^3 h}{\partial x_\alpha \partial x_\gamma \partial x_\gamma} = \frac{\partial}{\partial x_\alpha}\left(h\frac{\partial^2 h}{\partial x_\gamma^2}+\frac{1}{2}\left(\frac{\partial h}{\partial x_\gamma}\right)^2\right)-\frac{\partial}{\partial x_\gamma}\left(\frac{\partial h}{\partial x_\alpha}\frac{\partial h}{\partial x_\gamma}\right), \tag{2.11}$$

which gives the momentum equation in the form

$$\left.\begin{aligned} \frac{\partial}{\partial t}(hU_\alpha) &= -\frac{\partial}{\partial x_\beta}(hU_\alpha U_\beta+P_{\alpha\beta}), \\ \text{where}\qquad P_{\alpha\beta} &= \delta_{\alpha\beta}\left[\frac{gh^2}{2}+B-\frac{S}{\rho_w}\left\{h\frac{\partial^2 h}{\partial x_\alpha^2}+\frac{1}{2}\left(\frac{\partial h}{\partial x_\alpha}\right)^2\right\}\right]+\frac{S}{\rho_w}\frac{\partial h}{\partial x_\alpha}\frac{\partial h}{\partial x_\gamma}\delta_{\gamma\beta}. \end{aligned}\right\} \tag{2.12}$$

Equations (2.6) and (2.12) are the required shallow water equations. The term hU_α can be eliminated by cross-differentiation, and after some rearrangement, this leads to the equation,

$$\left.\begin{aligned} \frac{\partial^2 h}{\partial t^2}-gh_0\frac{\partial^2 h}{\partial x_\alpha^2} &= \frac{\partial^2 T_{\alpha\beta}}{\partial x_\alpha \partial x_\beta}, \\ \text{where}\qquad T_{\alpha\beta} &= hU_\alpha U_\beta+P_{\alpha\beta}-gh_0 h\delta_{\alpha\beta}. \end{aligned}\right\} \tag{2.13}$$

This equation is the two-dimensional inhomogeneous wave equation, and governs the generation and propagation of shallow water waves. It can be shown that at points in the wave field (for which the wave speed is $(gh_0)^{\frac{1}{2}}$) $T_{\alpha\beta}$ is zero to second order in the wave amplitude provided that the non-dispersive depth is chosen. Then the linear terms in B and $(S/\rho_w)\,h(\partial^2 h/\partial x_\alpha^2)$ exactly cancel. Also, in a region of convected turbulence, an order of magnitude analysis shows that the dominant term of $T_{\alpha\beta}$ is $hU_\alpha U_\beta$. The form of the shallow water equations, and these results about $T_{\alpha\beta}$, show a remarkable similarity to Lighthill's theory of sound, a similarity which might be exploited by modelling certain aerodynamic problems on a shallow layer of water. Accordingly, in the next section we seek a solution of (2.13) in exactly the same sense that the Lighthill–Curle (1955) equations represent a solution to the aerodynamic problem.

3. General theory

In the theory of aerodynamic sound developed by Lighthill (1952, 1954, 1962, 1963), the equations of motion of a compressible gas are written in the form

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho u_i) = 0, \tag{3.1}$$

$$\frac{\partial}{\partial t}(\rho u_i)+\frac{\partial}{\partial x_j}(\rho u_i u_j+p_{ij}) = 0. \tag{3.2}$$

ρ is the density, u_i the velocity component in the x_i direction, p_{ij} the compressive stress tensor, and the subscripts i, j, range over the values 1, 2 and 3. Lighthill combined these equations to yield the three-dimensional inhomogeneous wave equation,

$$\left.\begin{aligned}\frac{\partial^2 \rho}{\partial t^2} - a_0^2 \frac{\partial^2 \rho}{\partial x_i^2} &= \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j},\\ \text{where}\qquad T_{ij} &= \rho u_i u_j + p_{ij} - a_0^2 \rho \delta_{ij},\end{aligned}\right\} \tag{3.3}$$

and a_0 is the speed of sound in the gas at rest. This equation governs the generation and propagation of sound waves; it shows how the sound is equivalent to that generated by a volume distribution of quadrupoles of strength density T_{ij}. The influence of solid boundaries upon the sound was investigated by Curle (1955), who used the standard Kirchhoff solution of equation (3.3) to show how surface stresses are acoustically equivalent to a surface distribution of acoustic dipoles. Although Curle's result was only derived for finite surfaces, its more general validity is easily established.

The situation for shallow water is very similar. We have already seen how the equations of motion can be written in the form

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x_\alpha}(hU_\alpha) = 0, \tag{3.4}$$

$$\frac{\partial}{\partial t}(hU_\alpha) + \frac{\partial}{\partial x_\alpha}(hU_\alpha U_\beta + P_{\alpha\beta}) = 0, \tag{3.5}$$

which lead to the wave equation

$$\frac{\partial^2 h}{\partial t^2} - c^2 \frac{\partial^2 h}{\partial x_\alpha \partial x_\alpha} = \frac{\partial^2 T_{\alpha\beta}}{\partial x_\alpha \partial x_\beta}, \tag{3.6}$$

where $c^2 = gh_0$. To solve this equation, we observe that Lighthill's (1952) equation (3.3) reduces to it if ρ and T_{ij} are independent of the co-ordinate x_3. Curle's general solution of equation (3.3) expresses ρ in terms of volume and surface integrals of T_{ij} and p_{ij}. Consequently, for ρ to be independent of x_3, T_{ij}, p_{ij}, and the geometrical situation, must all be independent of x_3. We conclude that solutions of (3.6), in the presence of a contour Γ, are identical to the solutions of (3.3) in the presence of a cylinder S erected on Γ, and in which T_{ij} and p_{ij} do not depend on x_3. In this situation, if V denotes the volume exterior to S, then Curle's solution becomes

$$4\pi c^2(h(x_1, x_2, t) - h_0) = \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\int_V [T_{\alpha\beta}]\frac{dV(\mathbf{y})}{R} + \frac{\partial}{\partial x_\alpha}\int_S [P_\alpha]\,\frac{dS(\mathbf{y})}{R}. \tag{3.7}$$

R is the distance of the field point $\mathbf{x}$ from the source point $\mathbf{y}$ and is given by

$$R^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + y_3^2. \tag{3.8}$$

The square brackets imply the integrand is to be evaluated at the retarded time $t' = t - (R/c)$. P_α is written for $l_\beta P_{\alpha\beta}$, l_β being the direction cosines of the normal to Γ (and S).

As the only dependence of the integrands upon y_3 is through the retarded time, the y_3 integration is effectively a time integration. Accordingly, y_3 is replaced by t' as the independent variable, and (3.7) becomes

$$2\pi c^2(h(\mathbf{x},t)-h_0) = \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\int dA \int_{-\infty}^{t-r/c} \frac{T_{\alpha\beta}(\mathbf{y},t')\,c\,dt'}{[c^2(t-t')^2-r^2]^{\frac{1}{2}}} + \frac{\partial}{\partial x_\alpha}\int d\Gamma \int_{-\infty}^{t-r/c} \frac{P_\alpha(\mathbf{y},t')\,c\,dt'}{[c^2(t-t')^2-r^2]^{\frac{1}{2}}}. \qquad (3.9)$$

Here r is the two-dimensional radiation distance, given by

$$r^2 = (x_1-y_1)^2+(x_2-y_2)^2, \qquad (3.10)$$

and dA, $d\Gamma$ are two-dimensional area and line elements respectively. The contour integral is taken around Γ, and the area integral over the area external to Γ. This form of the solution has the advantage that no reference is made to the three-dimensional model used to generate it. This result can also be obtained from Volterra's solution of equation (3.6) (see Jones 1964, p. 42), but the given derivation is more straightforward.

Equation (3.9) is the fundamental result of the theory. It is an expression for the depth in terms of the known quantities $T_{\alpha\beta}$ and P_α, for any arbitrary fluid motion about a static solid surface. From the form of this equation, it follows that the waves are the same as those produced by quadrupole sources of strength density $T_{\alpha\beta}$, distributed over the region external to Γ plus dipole sources of strength P_α distributed around Γ. This equation also shows that the waves induced at any field point not only depend on the strength of the sources at a time r/c earlier, but on all previous times as well. On the other hand, the presence of the square root factor weights each contribution differently, and, since it is singular at $t' = t-r/c$, the main contribution comes from that region. In the far field, near these values of t', the square root can be approximated by $\{2r(ct-ct'-r)\}^{-\frac{1}{2}}$, and the time integral consequently yields an expression of the form $r^{-\frac{1}{2}}F(t-r/c)$. This implies that in the far field the waves from each source are waves of constant profile travelling at speed c, and whose amplitude falls off like $r^{-\frac{1}{2}}$. This type of behaviour is also found in linear theory for the conical wave field about a supersonic projectile (Whitham 1950). It is clearly so for sources of an oscillatory nature, where the method of stationary phase furnishes a precise form for F. We conclude that in the far field, water waves behave very similarly to sound waves, being produced by a similar distribution of dipoles and quadrupoles, and propagating in the same manner.

The result that the depth depends upon a time integral of $T_{\alpha\beta}$ or P_α merits further comment. It suggests that in any analysis featuring order of magnitude estimations, a typical time will be included. This does not usually happen in the three-dimensional theory. The typical time often varies with the parameters of the situation, and will result in a parametric dependence different from that obtained in three dimensions. Thus, some results of aerodynamic sound theory cannot be taken over directly to shallow water theory, but must be reconsidered in the light of (3.9). To illustrate this, the two-dimensional waves generated by a region of turbulence are examined in the next section. The corresponding three-dimensional theory is well known, being the basis of jet noise theory, so comparisons are easily made.

4. Two-dimensional waves generated by convected turbulence

Before the two-dimensional theory of waves produced by convected turbulence is developed, it is worthwhile briefly discussing the three-dimensional theory. This has been developed by Lighthill (1952, 1954, 1962, 1963) and Ffowcs Williams (1963). The latter considered the problem of a jet aircraft flying at a Mach number N, emitting a turbulent exhaust whose eddies move at Mach number M. He found that the mean square density fluctuation observed in the far field varies as

$$\overline{(\rho-\rho_0)^2} \sim \overline{\rho}^2 \frac{l^2}{R_0^2} M^7(M+N)|1+N\cos\phi|\,|1-M\cos\theta|^{-5}. \tag{4.1}$$

R_0 is the mean distance from the observer to the turbulence, l is the typical turbulence length scale, and θ, ϕ are angles specifying the direction of the convective motion. Along the lines $(1-M\cos\theta) = 0$, where the above result is not valid, a separate analysis gave the variation as

$$\overline{(\rho-\rho_0)^2} \sim \overline{\rho}^2 \frac{l^2}{R_0^2} M^2(M+N)|1+N\cos\phi|. \tag{4.2}$$

By assuming a particular form for the unknown correlation function which arose in his integral, Ffowcs Williams was able to evaluate it exactly, and deduced that for all M, the density fluctuation varies as

$$\overline{(\rho-\rho_0)^2} \sim \overline{\rho}^2 \frac{l^2}{R_0^2} M^7(M+N)|1+N\cos\phi|((1-M\cos\theta)^2+b^2M^2)^{-\frac{5}{2}}, \tag{4.3}$$

b being a small numerical constant. However, the general validity of (4.3) is not easily established, requiring detailed appeal to the theory of generalized functions. Nevertheless, a two-dimensional result similar to (4.3) can be obtained from Ffowcs Williams's equations, without recourse to such methods.

In deriving this two-dimensional result, the following convention is adopted. Vectors denoted by capital letters are three-dimensional vectors, whereas small letters denote vectors in the two-dimensional plane $X_3 = 0$. $\mathbf{k}$ denotes the unit vector normal to this plane. In his three-dimensional theory, Ffowcs Williams considers a region of turbulence which convects through space at a velocity $-a_0\mathbf{N}$, and which is composed of eddies travelling at a velocity $+a_0\mathbf{M}$. His equation (1.29) shows that the leading term of the mean square density fluctuation observed in the far field is given by the expression

$$\overline{(\rho-\rho_0)^2}(\mathbf{X},t) \sim \frac{1}{16\pi^2 a_0^8}\iint \frac{(X_i-Y_i)(X_j-Y_j)(X_k-Y_k)(X_l-Y_l)}{||\mathbf{X}-\mathbf{Y}|+\mathbf{N}.(\mathbf{X}-\mathbf{Y})|\,||\mathbf{X}-\mathbf{Y}|-\mathbf{M}.(\mathbf{X}-\mathbf{Y})|^5} \times \frac{\partial^4}{\partial\tau^4} P_{ijkl}(\mathbf{H},\mathbf{\Lambda},\tau)\, dV(\mathbf{H})\, dV(\mathbf{\Lambda}). \tag{4.4}$$

P_{ijkl} is the covariance of the stress tensor T_{ij}, $\mathbf{Y}$ is defined by the equation

$$\mathbf{Y} = \mathbf{H} - a_0\mathbf{N}t + \mathbf{N}|\mathbf{X}-\mathbf{Y}|, \tag{4.5}$$

and the two volume integrals are to be taken over the turbulent region. The retarded time τ is defined in terms of the other variables, and will be discussed at a later stage.

To obtain the two-dimensional result, this expression is applied to the situation in which there is no variation of the strength and geometry of the sources with X_3, and in which the vectors $\mathbf{N}$ and $\mathbf{M}$ are two-dimensional vectors $\mathbf{n}$ and $\mathbf{m}$. This use of the leading term of the three-dimensional expression to furnish the far field expression in two dimensions is valid as long as the predicted result is not zero. Because of the symmetry about the plane $X_3 = 0$, each volume integral is calculated over half the space, and the answer doubled. By introducing the simplifying notation $\mathbf{R} = (\mathbf{X}-\mathbf{Y})$, $\mathbf{r} = (\mathbf{x}-\mathbf{y})$, $R = |\mathbf{R}|$ and $r = |\mathbf{r}|$ (consequently $\mathbf{R} = \mathbf{r}+\mathbf{k}H_3$ and $R^2 = r^2+H_3^2$) expression (4.4) becomes in shallow water terminology

$$\overline{(h-h_0)^2}(\mathbf{x},t) \sim \frac{1}{4\pi^2c^8}\iint \frac{r_\alpha r_\beta r_\gamma r_\delta}{|R+\mathbf{n}.\mathbf{r}|\,|R-\mathbf{m}.\mathbf{r}|^5}\,\frac{\partial^4}{\partial\tau^4}P_{\alpha\beta\gamma\delta}\,(\boldsymbol{\eta},\boldsymbol{\lambda},\tau) \times dA(\boldsymbol{\eta})\,dA(\boldsymbol{\lambda})\,dH_3\,d\Lambda_3, \tag{4.6}$$

the range of integration for H_3 and Λ_3 is $(0, \infty)$.

The expression for the retarded time τ, given by Ffowcs Williams in his equation (1.29), was derived on the assumption that the eddy size was small compared with the radiation distance. Clearly, this approximation is not valid in the two-dimensional situation, where the eddies are infinite cylinders. In this case, the exact expression for τ (given in his equation (1.12)) can be approximated by

$$\tau = \frac{\boldsymbol{\lambda}.\mathbf{r}+\frac{1}{2}(H_3^2-\Lambda_3^2)}{c|R-\mathbf{m}.\mathbf{r}|}. \tag{4.7}$$

This expression is valid for all values of H_3 and Λ_3. The integrand only depends on Λ_3 through τ, and consequently the Λ_3 integral is effectively an integration over τ. Thus Λ_3 is replaced by τ as the independent variable, and expression (4.6) becomes,

$$\overline{(h-h_0)^2}(\mathbf{x},t) \sim \frac{1}{4\pi^2c^7}\iint \frac{r_\alpha r_\beta r_\gamma r_\delta}{|R+\mathbf{n}.\mathbf{r}|\,|R-\mathbf{m}.\mathbf{r}|^4}\,\frac{\partial^4}{\partial\tau^4}P_{\alpha\beta\gamma\delta}\,(\boldsymbol{\eta},\boldsymbol{\lambda},\tau) \times\frac{dA(\boldsymbol{\eta})\,dA(\boldsymbol{\lambda})\,dH_3\,d\tau}{\{H_3^2-2[c\tau|R-\mathbf{m}.\mathbf{r}|-\boldsymbol{\lambda}.\mathbf{r}]\}^{\frac{1}{2}}}. \tag{4.8}$$

The τ integral goes from $-\infty$ up to the zero of the square root.

A turbulent eddy is typically of spatial dimension l and life-time l/bcm, where $m = |\mathbf{m}|$ and b is a small numerical factor. As the covariance $P_{\alpha\beta\gamma\delta}(\boldsymbol{\eta},\boldsymbol{\lambda},\tau)$ is negligible unless $\boldsymbol{\lambda}$ and τ are within these ranges, the introduction of the scaled variables

$$\boldsymbol{\mu} = \frac{\boldsymbol{\lambda}}{l} \quad\text{and}\quad T = \frac{bcm}{l}\tau \tag{4.9}$$

reveals the dependence of the integral upon these scales. Furthermore, because this scaling places the significant values of $P_{\alpha\beta\gamma\delta}(\boldsymbol{\eta},\boldsymbol{\lambda},\tau)$ within constant and equal ranges of $\boldsymbol{\mu}$ and T, the transformed function $P'_{\alpha\beta\gamma\delta}(\boldsymbol{\eta},\boldsymbol{\mu},T)$ does not retain any major distinction between the new axes. Consequently, it is possible to rotate these axes without complicating the task of estimating the magnitude of $P'_{\alpha\beta\gamma\delta}(\boldsymbol{\eta},\boldsymbol{\mu},T)$, and this freedom to rotate axes allows the integral to be further simplified. The zero of the square root defines a plane in the $(\boldsymbol{\mu}, T)$ space, and the integration is to be carried out over the volume on one side of it. The rotated axes are chosen to be the normal to this plane, and any two axes parallel to it. The equation of the plane, in normal form, is

$$f^{-1}|R-\mathbf{m}.\mathbf{r}|T-f^{-1}bm\boldsymbol{\mu}.\mathbf{r}=f^{-1}\frac{bmH_3^2}{2l}, \tag{4.10}$$

where f is the normalizing factor $\{|R-\mathbf{m}.\mathbf{r}|^2+b^2m^2r^2\}^{\frac{1}{2}}$. If the new axes are denoted by ζ_1, ζ_2, ζ_3; ζ_3 being the normal axis, then ζ_3 is given by

$$\zeta_3=f^{-1}|R-\mathbf{m}.\mathbf{r}|T-f^{-1}bm\boldsymbol{\mu}.\mathbf{r}. \tag{4.11}$$

The derivatives are related by

$$\frac{\partial}{\partial T}=a_1\frac{\partial}{\partial\zeta_1}+a_2\frac{\partial}{\partial\zeta_2}+f^{-1}|R-\mathbf{m}.\mathbf{r}|\frac{\partial}{\partial\zeta_3}, \tag{4.12}$$

a_1, a_2 being factors determined by the particular choice of ζ_1 and ζ_2. Knowledge of these is not necessary, since they vanish after integration over ζ_1 and ζ_2. Equation (4.8) reduces to the simpler form,

$$\overline{(h-h_0)^2}(\mathbf{x},t)\sim\frac{b^3m^3}{4\pi^2c^4l}\iint\frac{r_\alpha r_\beta r_\gamma r_\delta}{|R+\mathbf{n}.\mathbf{r}|f^4}\frac{\partial^4P'_{\alpha\beta\gamma\delta}}{\partial\zeta_3^4}(\boldsymbol{\eta},\boldsymbol{\zeta})\frac{dA(\boldsymbol{\eta})\,dH_3\,dV(\boldsymbol{\zeta})}{\left[H_3^2-\dfrac{2lf}{bm}\zeta_3\right]^{\frac{1}{2}}}. \tag{4.13}$$

For values of H_3 such that $H_3^2\gg 2lf/bm$, the square root in the denominator can be approximated by H_3. These values of H_3 contribute nothing to the integral, since after this approximation, the ζ_3 integral goes to zero. We conclude that only values of H_3 such that $H_3^2\sim 2lf/bm$ contribute to the integral. For large values of r, this contribution range only increases as $(lr)^{\frac{1}{2}}$, and as $R^2=r^2+H_3^2$, it follows that R can be approximated by r in the far field. If H_3 is scaled by the factor $2lf^{\frac{1}{2}}/bm$ to standardize its contribution range, then (4.13) becomes

$$\overline{(h-h_0)^2}(\mathbf{x},t)\sim\frac{b^3m^3}{4\pi^2c^4l}\iint\frac{r_\alpha r_\beta r_\gamma r_\delta}{|r+\mathbf{n}.\mathbf{r}|((r-\mathbf{m}.\mathbf{r})^2+b^2m^2r^2)^2}\frac{\partial^4P'_{\alpha\beta\gamma\delta}}{\partial\zeta_3^4}(\boldsymbol{\eta},\boldsymbol{\zeta})\times\frac{dA(\boldsymbol{\eta})\,dH_3'\,dV(\boldsymbol{\zeta})}{(H_3'^2-\zeta_3)^{\frac{1}{2}}}. \tag{4.14}$$

The parametric variation of $\overline{(h-h_0)^2}$ quickly follows from this equation. As $P'_{\alpha\beta\gamma\delta}(\boldsymbol{\eta},\boldsymbol{\zeta})$ is a mean square of the tensor $T_{\alpha\beta}$, it varies as $h^2c^4m^4$. The $\boldsymbol{\eta}$ area integral yields a typical source area, which varies as $l^2([m+n]/m)$. If $\mathbf{m}.\mathbf{r}$ and $\mathbf{n}.\mathbf{r}$ are written as $mr\cos\theta$ and $nr\cos\phi$ respectively, then the two-dimensional result equivalent to (4.3) is

$$\overline{(h-h_0)^2}\sim\overline{h^2}\frac{l}{r_0}m^6(m+n)|1+n\cos\phi|((1-m\cos\theta)^2+b^2m^2)^{-2}. \tag{4.15}$$

This result differs from the three-dimensional result (4.3) in two respects. First, the inverse square law of sound intensity is replaced by a first power law, as is to be expected upon considerations of energy. Secondly, the 'Lighthill eighth power law' is here replaced by a seventh power law,† coupled with a corresponding change in the directional factor. This is entirely due to the infinite

† *Note added in proof.* This result has also been found by Obermeier (1967) in his study of two-dimensional aerodynamic sound.

length of the eddies, which makes retarded time differences crucial in determining the effective volume of each eddy. This contrasts with the sound theory, where such time differences are usually unimportant in this respect.

5. Conclusion

The main conclusion to be drawn from this analysis is that qualitatively, water waves behave in a very similar manner to sound waves. Both radiate out from their sources at a constant speed, preserving their profile in the far field. Furthermore, they are generated by the same equivalent system of dipoles and quadrupoles. As a consequence of this, for both types of waves, a region of turbulence generates a directional field, whose intensity varies as a high power of the eddy convective speed. The analysis shows that quantitative results are slightly different for the shallow water waves, the intensity increasing with the seventh power of flow velocity and the fourth power of the Doppler factor $(1-m\cos\theta)^{-1}$. Though a peak is found at the Mach wave condition, no singularity is evident, even in the first approximation which is then valid at all speeds. The similarity of this result with recent developments in the theory of aerodynamic sound generation leads to the possibility that turbulence generated shallow water waves can form a satisfactory and easily visualized simulation of aerodynamic noise problems of a rather intractible kind. Indeed some experiments have already been attempted and the qualitative similarity with the aerodynamic problem is very evident. That there should also be a means of making the similarity quantitative is the main outcome of this work, though it should be emphasized that certain properties of the two-dimensional wave field distort any complete analogy with the three-dimensional problem.

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